# The pressure distribution on dihedral wings at supersonic speed

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For wings with supersonic edges and with arbitrary dihedral, twist, camber and thickness distribution, the pressure distribution on the wing exterior to and along the two Mach lines emanating from the vertex of the wing is equal to the corresponding pressure distribution for a planar wing. The problem is to find the pressure distribution inside the two Mach lines. In the present paper, the unknown pressure distribution is approximated by an elementary function of the two surface variables. The (as yet undetermined) constants in the function are then found by the conditions: (i) that the function takes on the corresponding planar values along the two Mach lines, (ii) that it fulfils certain generalized integral relationships (Ting 1959), and (iii) that it satisfies the averaging property of solutions of the wave equation to be developed in this paper. The generalized integral relationship relates the integral of the pressure distribution along the line of intersection of a Mach plane with the wing to the integral along the same line of the prescribed normal velocity. The averaging property relates the pressure distribution along the line of intersection of the surface of the dihedral wing to that on a planar wing.

## 1. Introduction

The problem of linearized supersonic flow over thin wings with an arbitrary dihedral can be solved if the flow field can be resolved into conical fields (Snow 1948; Germain 1955). On the other hand, if the normal velocity on the wing surface is arbitrary while the dihedral angle is equal to 90°, the problem can be treated by the method of images (Ting 1957).

In general, the problem of supersonic flow over thin wings with supersonic edges, arbitrary dihedral, and with arbitrary twist, camber and thickness distribution, has been solved from a mathematical point of view. However, the numerical evaluation of the solutions is extremely difficult. Moreover, the problem of wings with dihedral is the essence of the interference problem associated with wings and prismatic bodies. Recent studies by Ferri (1955, p. 353), Ferri & Clark (1957), and Ferri, Clarke & Ting (1957) on the effect of body contouring for drag reduction at constant lift have increased the interest in solutions of wing-body interference problems and in the determination of the pressure distribution on dihedral wings.

This paper begins with the derivation of an averaging property of solutions of the wave equation. The integral relationship derived by Ting (1959) as a generalization of the results of Lagerstrom & Van Dyke (1949), Bleviss (1953), Ferri (1955) and Ferri *et al.* (1957) is then restated and applied to dihedral problems.

When the flow field is homogeneous, as defined in §4, the procedure for obtaining the approximate pressure distribution on the wing is outlined. Numerical examples are provided for flows whose potentials are homogeneous of order 1 and 2. The former is identical to conical flow, and the corresponding solution is in good agreement with the exact solution of the linearized conical problem.

When the flow field cannot be resolved into a finite number of homogeneous flow fields, a more extended procedure for obtaining the pressure distribution is required; this is outlined and illustrated by an example.

## 2. Averaging property of solutions of the wave equation

This can be described clearly by referring to the following specific physical problem. For supersonic flow over dihedral wings with prescribed normal velocity on the two planes of the wing, the product of the dihedral angle and the pressure along the line of intersection of the planes is independent of the dihedral angle, and is therefore equal to $\pi$ times the corresponding pressure distribution for the planar wing.

The mathematical formulation and the proof of this statement are given as follows:

Let $\phi^\nu$ be the solution of the wave equation

$$\phi_{yy} + \phi_{zz} - \phi_{tt} = 0, \tag{1}$$

which fulfils the initial conditions

$$\phi^\nu = 0 \quad \text{and} \quad \phi^\nu_t = 0 \quad \text{at} \quad t = 0, \tag{2}$$

and satisfies the boundary conditions for the normal derivative of $\phi^\nu$, namely

$$\phi^\nu_n = g_1(z, t) \tag{3}$$

on the $t$-$z$ plane where $t > 0$, $z > 0$, and

$$\phi^\nu_n = g_2(z, t) \tag{4}$$

on the $s$-$t$ plane defined by the parametric equations, $z = s\cos\nu$ and $y = s\sin\nu$, where $s > 0$ and $t > 0$ (cf. figure 1).

The superscript $(\nu)$ of $\phi^\nu$ is associated with the angle between the $t$-$z$ plane and the $s$-$t$ plane. Then, for different values of $\nu$, the corresponding solutions on the line of intersection of the boundary planes obey the following relationship:

$$\nu\phi^\nu(0, 0, t) = \pi\phi^\pi(0, 0, t). \tag{5}$$

Equation (5) can be proved by virtue of Green's Theorem (see Ward 1955, p. 55), which states

$$\iint_A^* \phi^\nu(y, z, t)\, \mathbf{Q}.\mathbf{n}\, dA - \iint_A^* (\mathbf{P}.\mathbf{n})\,(1/R)\, dA = 0, \tag{6}$$

where (*) means 'the finite part of', and

$$R = [(t_1-t)^2-z^2-y^2]^{\frac{1}{2}},$$

$$\mathbf{P} = \left(\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}-\mathbf{i}\frac{\partial}{\partial t}\right)\phi^\nu,$$

$$\mathbf{Q} = \left(\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}-\mathbf{i}\frac{\partial}{\partial t}\right)(1/R).$$

**i**, **j**, **k** are the unit vectors along the $t$, $y$ and $z$ axes respectively, and **n** is the unit outward normal vector to the closed surface of integration $A$. The surface $A$ consists of (see figure 1):

(i) $A^*$, which is the part of the surface $R = 0$ lying in between the planes $t = 0$, $t = t_1-\delta$ and the boundary planes.

(ii) I, which is the part of the plane $t = 0$, lying inside $A^*$.

(iii) II, which is the part of the plane $t = t_1-\delta$, lying inside $A^*$.

(iv) $B_1$, which is the part of the $t$-$z$ plane, lying inside $A^*$ and between planes $t = 0$ and $t = t_1-\delta$.

(v) $B_2$, which is the part of the $t$-$s$ plane, lying inside $A^*$ and between planes $t = 0$ and $t = t_1-\delta$.

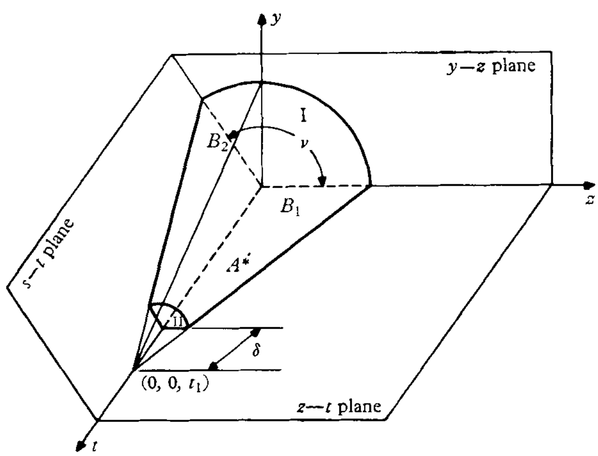


FIGURE 1. Averaging property with boundary data.

As $\delta \to 0$, the finite part of the first integral over II approaches $\nu\phi^\nu(0,0,t_1)$ and the second integral over II approaches zero and that of both integrals over $A^*$ vanishes (see Ward 1955).

Due to the initial conditions, the boundary conditions, and the vanishing of $\mathbf{Q}.\mathbf{n}$ on the $t$-$z$ and $t$-$s$ planes, equation (6) becomes:

$$\nu\phi^\nu(0,0,t_1) = -\iint_{B_1}\frac{g_1(z,t)\,dz\,dt}{\sqrt{[(t_1-t)^2-z^2]}}-\iint_{B_2}\frac{g_2(s,t)\,ds\,dt}{\sqrt{[(t_1-t)^2-s^2]}}. \tag{7}$$

The terms on the right-hand side of equation (7) are independent of $\nu$, and therefore $\nu\phi^\nu$ is independent of $\nu$ and equation (5) is valid.

This property is formulated with homogeneous initial conditions and non-homogeneous boundary conditions. It will be called the averaging property associated with boundary data in order to differentiate from the averaging property associated with initial data (Ting 1958). The application of the second averaging property can be found in diffraction problems and in the interference problem of a wing and prismatic body.

## 3. The generalized integral relationship

In this section the requirement for and statement of the generalized integral relationship given by Ting (1959) are restated. As shown in figure 2, a cylindrical surface $y = F(z)$, is placed in a supersonic stream directed along the $x$-axis, with velocity $U$ and Mach number $M$. $q_n[x, F(z), z]$ represents the prescribed normal velocity on the cylindrical surface with

$$q_n/U \ll 1 \tag{8}$$

and

$$q_n[x, F(z), z] = 0 \quad \text{for} \quad x < 0. \tag{9}$$

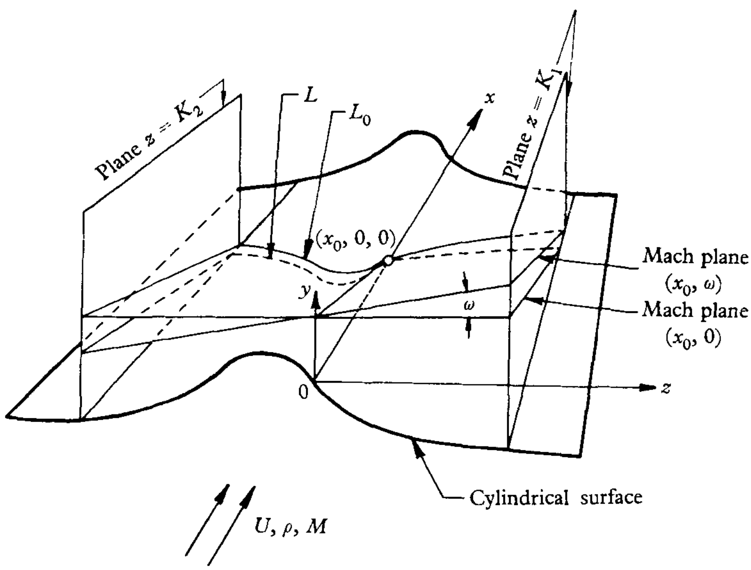


FIGURE 2. The generalized integral relationship.

It will be assumed that $q_n[x, F(z), z]$ is piece-wise continuous and that for any given Mach plane, $x + By = M\beta$, there exist two numbers $K_1(\beta)$ and $K_2(\beta)$ (with $K_1 > K_2$) such that on the Mach plane, the disturbance potential $\phi$, due to $q_n$, is confined inside the region $K_1 > z > K_2$. That is,

$$\phi(x, y, z) = 0$$

for

$$x \leqslant M\beta - By, \quad \infty > z \geqslant K_1(\beta) \quad \text{and} \quad K_2(\beta) \geqslant z > -\infty. \tag{10}$$

The integral of the disturbance pressure along the curve of intersection, $L_0$, of the cylindrical surface $y = F(z)$ with the Mach plane $x + By = M\beta$ is related to the integral of the prescribed normal velocity $q_n[x, F(z), z]$ as follows:

$$\int_{L_0} p \sin\sigma \, \mathbf{n}.\mathbf{j} \, dL_0 = \frac{\rho U}{B} \int_{L_0} q_n \sin\sigma \, dL_0, \tag{11}$$

where $\mathbf{n}$ is the unit vector normal to the cylindrical surface, $\mathbf{j}$ is the unit vector parallel to $y$-axis, and $\sigma$ represents the angle between the $x$-axis and the tangent to the path of integration.

The integral relationship still holds if the cylindrical surface is rotated about its generatrix, the $x$-axis, with respect to the co-ordinate axes and the Mach plane. Consequently, if the co-ordinate axes are fixed with respect to the cylindrical surface while the Mach plane is rotated about the $x$-axis, the integral relationship remains valid and equation (11) becomes

$$\int_L p \sin\sigma\, \mathbf{n}.\boldsymbol{\omega}\, dL = \frac{\rho U}{B} \int_L q_n \sin\sigma\, dL, \tag{12}$$

where $L$ refers to the line of intersection of the cylindrical surface with the Mach plane defined by

$$x + B(y\cos\omega + z\sin\omega) = M\beta = x_0 \tag{13}$$

and

$$\boldsymbol{\omega} = \mathbf{j}\cos\omega + \mathbf{k}\sin\omega.$$

The Mach plane, which is intercepted by the $x$-axis at the Mach angle, is specified by two parameters, the $x$ intercept $x_0$ and the angle of rotation $\omega$. For each value of $x_0$ and $\omega$, equation (12) gives a linear relationship between the pressure distribution and the normal velocity. It should be noted that for the derivation of the integral relationship the assumption of equation (10) has been made. For the fulfilment of equation (10), there exist certain restrictions on the values of two parameters, $x_0$ and $\omega$, and a rule for selecting the suitable cylindrical surface.

For dihedral wings with supersonic edges, it is sufficient to discuss only the upper surface for the case where the normal velocity on one of the dihedral planes, $y - z\tan\nu = 0$, vanishes. On the other plane, $y = 0$, the normal velocity $\phi_y = g(x, z > 0)$ is prescribed and vanishes for the region $x < 0$.

To apply the integral relationship, it is clear that the intercept $x_0$ should not be outside the wing surface. The value of $\omega$ should obey the restriction, $\frac{3}{2}\pi - \nu > \omega > -\frac{1}{2}\pi$.

For $\omega > 0$, the wing surface can be chosen as the cylindrical surface and the condition of equation (10) is fulfilled with $K_1 = M\beta/(B\sin\omega)$ and $K_2 = -M\beta/[B(1-\sin\omega)]$.

For $\omega < 0$, the Mach plane will not intercept the positive $z$-axis. In order to fulfil the condition of equation (10) a modified cylindrical surface should be used. It may be composed of (see figure 3): (1) the portion of the wing surfaces where $z < K'$, (2) the portion of the plane $z = K'$, where $0 < y < K'$, and (3) the portion of the plane $y = K'$, where $z > K'$. Here $K'$ is the $z$-co-ordinate of the point where the Mach plane intercepts the forward Mach line in the $x$-$z$ plane issuing from the vertex.

On the portion of the plane $y = K'$, where $z > K'$ and which is ahead of the Mach plane, the flow field is undisturbed; therefore, $K'$ can be chosen as the value for $K_1(\beta)$ in equation (10). The path of integration $L$ with non-zero integrand will then consist of two parts: (1) the segments of straight lines on the surface of the wing inside the domain of influence of the vertex, and (2) the segment of straight line on the plane $z = K'$ which is outside of the domain of influence of the vertex.

On the second segment the pressure distribution and the normal velocity can be obtained from the planar solutions; therefore, the integral relationship along the path $L$ does not involve any unknown function other than the pressure distribution on the wing surface. This is the rule to be observed in selecting the proper cylindrical surface.

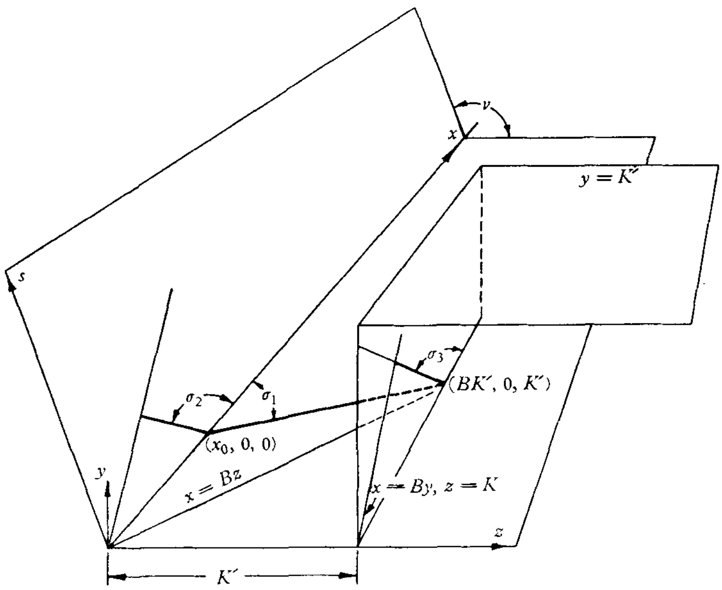


FIGURE 3. Integral relationship for dihedral wings—when the Mach plane does not intercept the positive $z$-axis.

## 4. Homogeneous flow

It is sufficient to discuss the case $q_n = 0$ on the dihedral plane $y = z \tan \nu$. On the other plane $y = 0$, the prescribed normal velocity is now expressed as

$$v\,(x > 0, 0, z > 0) = x^m z^n, \tag{14}$$

where $m$ and $n$ are real numbers. The flow field is a homogeneous flow of the $(m+n+1)$th order (see Germain 1955) for which an analytic solution other than the conical solution (homogeneous flow of order 1) is not yet available. However, the method of the present paper can be readily applied regardless of the order.

The pressure distribution on the wing surface inside the domain of influence of the vertex can be expressed in general as

$$\frac{B}{\rho U^2} p_{m,n}(x, 0, z) = x^{m+n} f_1(\eta_1), \tag{15a}$$

and

$$\frac{B}{\rho U^2} p_{m,n}(x, s \sin \nu, s \cos \nu) = x^{m+n} f_2(\eta_2), \tag{15b}$$

where

$$\eta_1 = \frac{Bz}{x} \quad \text{and} \quad \eta_2 = \frac{Bs}{x}.$$

If the functions $f_1$ and $f_2$ are approximated by second-order polynomials, i.e.

$$f_j(\eta_j) = a_{0j} + a_{1j}\eta_j + a_{2j}\eta_j^2. \tag{16}$$

there are altogether six unknown constants $a_{0j}$, $a_{1j}$ and $a_{2j}$, where $j = 1, 2$. The condition along the Mach lines emanating from the vertex yields the values of $f_1(1)$ and $f_2(1)$, while the averaging property yields $f_1(0)$ and $f_2(0)$. The remaining two linear equations for the six unknowns are furnished by the integral relationship corresponding to two different values of $\omega$. It is evident that the integral relationship is independent of the $x$-intercept, $x_0$, of the Mach plane.

For the homogeneous flow of order 1 ($m = n = 0$), the constants are determined for $\nu = 135°$ and the pressure distribution is plotted in figure 4. The deviation from the exact solution of the linearized conical problem is within 10%.

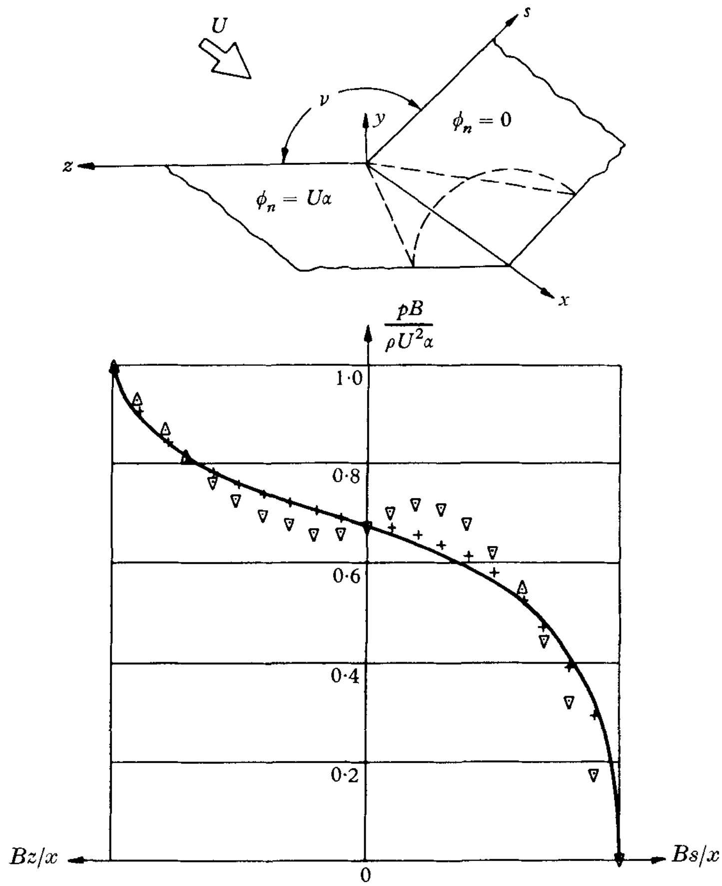


FIGURE 4. Homogeneous flow field of order 1. ——, exact linearized conical solution; +, approximate solution of equation (17); $\nabla$, approximate solution of equation (16).

With the knowledge that the expression for the pressure distribution contains a term like $\sqrt{(1-\eta_j^2)}$ near the Mach lines, it is, therefore, desirable to admit such a term in the approximate expressions of $f_j(\eta_j)$; e.g.

$$f_j(\eta_j) = a_{0j} + a_{1j}\eta_j + b_j\sqrt{(1-\eta_j^2)}. \tag{17}$$

The six unknown constants $a_{0j}$, $a_{1j}$ and $b_j$ are again determined for the homogeneous solution of order 1, and the corresponding pressure distribution is in good agreement with the exact linearized conical solution (figure 4).

A calculation has been made by Ting (1958) to determine the eight constants when the term $b_j\sqrt{(1-\eta_j^2)}$ is added to the right-hand side of equation (16). The pressure distribution is then in perfect agreement with the exact conical solution.

Figure 5 shows the pressure distribution on the wing for a homogeneous flow of order 2 with $m = 0$ and $n = 1$. The approximate solution in the form of equation (17) differs slightly from that of equation (16).

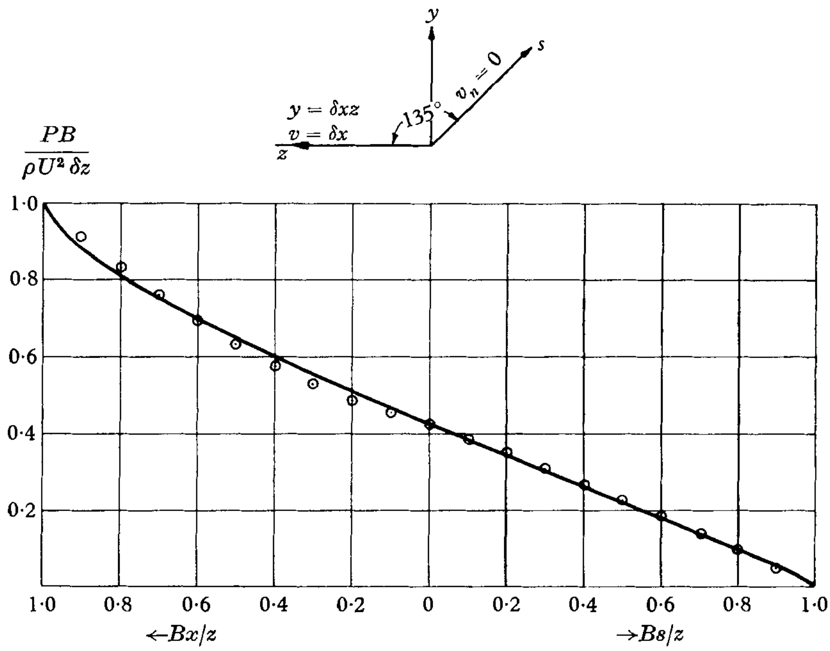


FIGURE 5. Homogeneous flow field of order 2. ——, approximate solution of equation (17); ⊙, approximate solution of equation (16).

## 5. An example of a non-homogeneous flow field

In principle, the prescribed normal velocity on the wing surface can be expressed as a double power series of $x$ and $z$. Consequently, the pressure distribution will be

$$p = \sum_m \sum_n a_{mn} p_{m,n}, \tag{18}$$

where $p_{m,n}$ can be obtained by the procedure outlined in the preceding section.

From the engineering point of view, this is not a practical solution if the series of equation (18) does not converge fast enough. For such cases, a different procedure of solution is necessary; this can be best illustrated by an example.

Figure 6 shows a dihedral wing with $\phi_n = 0$ on the plane $y = z \tan \nu$. On the plane $y = 0$, the normal velocity is

$$\phi_y = 0 \quad \text{for} \quad 0 < z < a \quad \text{or} \quad x < 0, \tag{19a}$$

$$\phi_y = \alpha U \quad \text{for} \quad z > a \quad \text{and} \quad x > 0. \tag{19b}$$

This problem is chosen because it exhibits the basic character of diffraction problems and also that of interference problems of wings with prismatic bodies.

For $x < aB$, the disturbance pressure is identical with the planar solution. At $x = aB$, the 'incident disturbance' $p^{(\pi)}$ which originates at the point $x = 0, y = 0, z = a$ reaches the $x$-axis. For $x > aB$, the incident disturbance is diffracted and/or reflected by the $x$-$s$ plane ($y = s \sin \nu$, $z = s \cos \nu$). The disturbance due to the corner is confined inside the forward Mach lines $x - Ba = Bz$ in the plane $y = 0$ and $x - Ba = Bs$ in the $x$-$s$ plane.

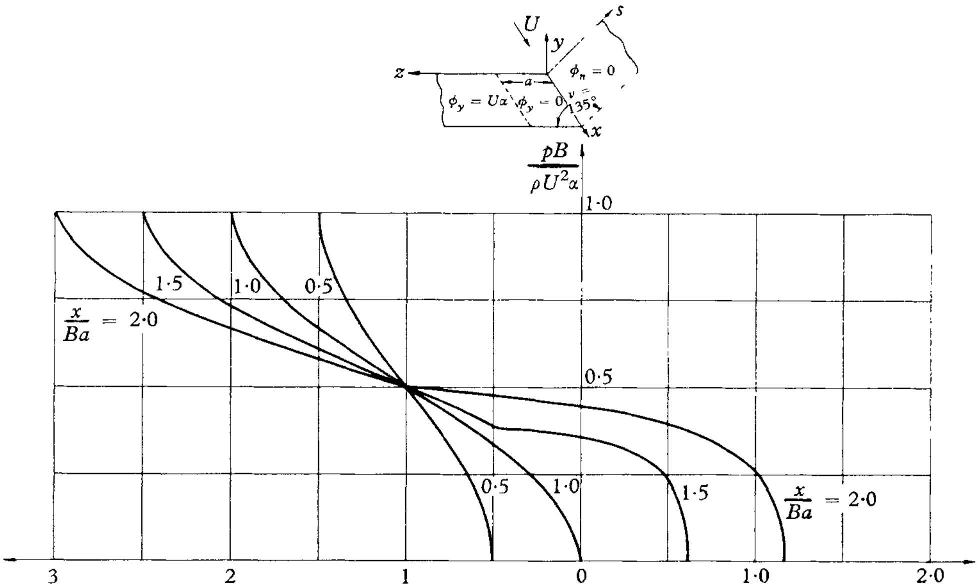


FIGURE 6. An example for a non-homogeneous flow field.

For the case where $\nu < \pi$, the incident disturbance is simply reflected by the $x$-$s$ plane in the region upstream of the Mach line $x - Ba = Bs$ and behind the hyperbola $x = B\sqrt{(s^2 - 2as \cos \nu + a^2)}$. In this region, the resultant pressure should be $2p^{(\pi)}(x, y = s \sin \nu, z = s \cos \nu)$. The pressure distribution in the domain of disturbance of the corner can be written as

$$p(x, 0, z) = p^{(\pi)}(x, 0, z) + p_1(\bar{x}, z), \tag{20a}$$

$$p(x, s \sin \nu, s \cos \nu) = 2p^{(\pi)}(x, s \sin \nu, s \cos \nu) + p_2(\bar{x}, s), \tag{20b†}$$

where $\bar{x} = x - Ba$, and $p_1$ and $p_2$ denote the deviations from the planar pressure distribution.

The conditions for $p_1$ and $p_2$ along the Mach lines are

$$p_1(Bz, z) = 0, \tag{21a}$$

and

$$p_2(Bs, s) = 0. \tag{21b}$$

The averaging property yields

$$p_1(\bar{x}, 0) = \left(\frac{\pi}{\nu} - 1\right) p^{(\pi)}(\bar{x} + Ba, 0, 0), \tag{22a}$$

and

$$p_2(\bar{x}, 0) = \left(\frac{\pi}{\nu} - 2\right) p^{(\pi)}(\bar{x} + Ba, 0, 0). \tag{22b}$$

† The term $2p^{(\pi)}$ should be omitted from equation (20b) for the case where $\nu > \pi$.

The unknown functions, $p_1$ and $p_2$, will be approximated by the following functions of two variables:

$$p_j = [a_{0j} + d_{0j}\eta_j + b_{0j}\sqrt{(1-\eta_j^2)}]\,[\bar{x}/(aB)]^{\frac{1}{2}} + [a_{1j} + d_{1j}\eta_j + b_{1j}\sqrt{(1-\eta_j^2)}]\,[\bar{x}/(aB)]^{\frac{3}{2}}, \tag{23}$$

where $j = 1, 2$, $\eta_1 = Bz/\bar{x}$ and $\eta_2 = Bs/\bar{x}$.

In the approximating functions, a total of twelve unknown constants have been admitted. Eight equations are obtained from equations (21) and (22) for $\bar{x} = 0{\cdot}5Ba$ and $Ba$, respectively, since in the present problem the length of the chord has been chosen to be $2Ba$. The integral relationship yields four equations corresponding to the Mach planes $(x_0, \omega)$ with $x_0 = Ba$, $x_0 = 0{\cdot}5Ba$ and $\omega = 90°$, $\omega = 45°$, respectively, for the case of $\nu = 135°$. The numerical results are shown in figure 6.

$p_j$ is expressed in a series of $[\bar{x}/(aB)]^{\frac{1}{2}}$, $[\bar{x}/(aB)]^{\frac{3}{2}}, \ldots$ in equation (23) due to the fact that $p^{(\pi)}(\bar{x}+aB, 0, 0)$ can be expressed by a similar series at least for small values of $\bar{x}/(aB)$. However, if a regular power series is employed, the result of the approximation does not differ much from that of equation (23).

For the purpose of improving the approximation, additional terms may be admitted. If terms involving higher powers of $\bar{x}/(aB)$ are added to equation (23), extra values of $x_0$ should be selected so that there will be additional integral relationships corresponding to the Mach plane $(x_0, \omega)$. On the other hand, if additional terms involving $\eta$ are admitted, the additional equations are obtained by taking more values of $\omega$.

## 6. Concluding remarks

The averaging property of solutions of the wave equation and the generalized integral relationship are applied to obtain the pressure distribution on dihedral wings if the normal velocity on the dihedral planes is prescribed.

Since the integral relationship gives a linear relationship between the integral of pressure and the integral of normal velocity, it is plausible to apply the integral relationship to solve the problem when the boundary conditions on the dihedral planes are of the mixed type. Consequently, it is feasible to extend the method in the present paper to solve the problem of dihedral wings with subsonic edges.

As a further extension of the work reported here, the problem wherein the boundary is a cylindrical surface may be cited. In this case it is necessary to obtain the pressure distribution based on the integral relationship alone. The feasibility of so obtaining the pressure distribution was demonstrated for the problem of conical flow on dihedral wings by Ting (1958). With the averaging condition replaced by an additional integral relationship, the approximate pressure distribution on the wing surface differs from that of the exact linearized conical solution by 2 %. Attempts are in progress to obtain, from the integral relationship alone, the pressure distribution on a cylindrical surface with its generator parallel to the direction of flow and with a prescribed normal velocity.

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